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Wall effects on electron dynamics in the SOL

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Abstract

In a tokamak divertor, steep gradients in the density and temperature profiles are usually established with a scale length that is often shorter than the mean-free path of energetic electrons. The electron flux is then non-local in nature, being determined not only by local density and temperature gradients but by integrals of these quantities over an extended region in space. In the present work, such non-local expressions for the electron particle and heat fluxes are derived, generalizing earlier work to include the effects of walls bounding the plasma. The results are found to be in excellent agreement with a numerical solution of the full kinetic equation, if the temperature falls off rapidly near the wall. In the opposite limit of a nearly isothermal plasma, a comparison with a variational calculation of the particle flux is presented, showing less satisfying agreement.

Keywords: Non-local transport; Analytical model; Kinetic analysis

1. Introduction

The Spitzer–Härm expression for the electron heat conductivity [1,2] becomes invalid already at quite small values of the mean-free path of thermal particles λ (when it exceeds $2 \cdot 10^{-3}$ times the macroscopic scale length L) [3]. The reason is that the energy is mostly carried by quite energetic electrons, whose mean-free path is much longer λ. On the other hand, plasmas in which $2 \cdot 10^{-3} < \lambda/L < 1$ are quite common. This has prompted a large number of attempts to generalize the Spitzer–Härm formula to include plasmas in this range of λ/L [4–7]. The heat flux is then not determined by the local temperature and density gradients alone, but becomes non-local in character. A particularly successful approach to this problem is furnished by the high Z approximation [5–7]: If the ion charge Z is taken to be sufficiently large, pitch-angle scattering is the dominant collisional process. As a result, the distribution function becomes nearly isotropic, and the electrons diffuse rather than stream freely along the magnetic field lines. Consequently, the kinetic equation can be simplified, and with additional simplifications it can be solved to yield non-local expressions for the heat flux, which sometimes compare favorably with more accurate, numerical results. Such simple analytical expressions are of great value for, e.g., implementation in codes simulating the edge plasma in tokamaks. The tokamak scrape-off layer is generally in the above mentioned range of λ/L, so the electron dynamics is usually governed by non-local effects.

This problem is well recognized in the edge plasma community. In numerical edge simulations, artificial flux-limits are usually employed to prevent the conductive heat flux to exceed the free-streaming value. However, by their very nature, flux limits cannot properly describe the opposite situation, when the heat flux *exceeds* the Spitzer value. This is common in the tokamak divertor, where fast electrons from the midplane may dominate the heat flux, as observed directly in kinetic simulations [8], and modelled phenomenologically by Cohen and Rognlien [9]. It is also widely acknowledged that non-local electron transport is of significance for the interpretation of probe measurements in the tokamak divertor [8,10].

It is the purpose of the present paper to compare the earlier non-local formula of Krasheninnikov [7] for electron particle and heat fluxes with a numerical solution of

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the full kinetic equation for fast electrons, and to generalize it to account for effects of material walls bounding the plasma. The existence of boundaries has been ignored in earlier treatments of non-local heat conduction (other than [9]), but is likely to be important in tokamaks since the divertor plates are usually situated near the region where temperature and density gradients are steep. The effect of the walls are described by new terms in the expressions for the heat and particle fluxes, which are calculated in Section 2, and persist even in a homogeneous background plasma. This enables a comparison with a more exact analytical solution found in Section 3. To our knowledge, it is the first analytic, kinetic treatment of this taking collisions consistently into account. In the last section our conclusions are summarized.

2. Non-local expressions for particle and heat fluxes

The kinetic equation for energetic electrons in one dimension is

$$\chi v \frac{\partial f}{\partial x} - \chi \frac{eE}{m} \frac{\partial f}{\partial v} - \frac{eE}{mv}(1-\chi^2)\frac{\partial f}{\partial \chi} = \frac{4\pi e^4 \Lambda n}{m^2}\left[\frac{1+Z}{v^3}\frac{\partial}{\partial \chi}(1-\chi^2)\frac{\partial f}{\partial \chi} + \frac{m}{v}\frac{\partial}{\partial \varepsilon}\left(f + T\frac{\partial f}{\partial \varepsilon}\right)\right] \quad (1)$$

where f is the distribution function, v the velocity, χv its component in the direction of the coordinate x (usually parallel to the magnetic field), m the electron mass, Ze and $-e$ the ion and electron charges, respectively, T and n the electron temperature and density, Λ the Coulomb logarithm, and ε denotes the sum of the kinetic and potential energies: $\varepsilon = mv^2/2 - e\phi$, with ϕ the electrostatic potential. If $Z \gg 1$, the electrons are frequently scattered according to the first term on the right-hand side, and a random walk takes place along x. Eq. (1) can thus be expected to reduce to a diffusion equation in this limit. Indeed, ordering the consecutive terms as $\delta:\delta^2:\delta^2:1:\delta^2$, where $\delta \ll 1$ is an expansion parameter, and expanding the distribution function, $f = f_0 + f_1 + f_2 \ldots$, gives [5,7]

$$\frac{\partial^2 f_0}{\partial y^2} + \frac{1}{\varepsilon^3}\frac{\partial}{\partial \varepsilon}\left[f_0 + T(y)\frac{\partial f_0}{\partial \varepsilon}\right] = 0, \quad (2)$$

where $\mathrm{d}y \equiv [6(Z+1)]^{1/2}\pi e^4 \Lambda n(x)\mathrm{d}x$. Here, we have used $\partial f_0/\partial \chi = 0$ from the zeroth-order part of Eq. (1), and

$$f_1 = -\chi\sqrt{\frac{6}{Z+1}}\,\varepsilon^2 \frac{\partial f_0}{\partial y} \quad (3)$$

from the first-order equation, and taken the average over χ in the next-order equation. In addition, the assumption $\varepsilon \gg T$ (so that f_0 varies much faster with y than do $n(y)$ or $T(y)$), already used in Eq. (1), is made to arrive at Eq. (2). The electrostatic potential ϕ is assumed to be of the same order as T/e.

In Ref. [7], an approximate solution to the kinetic Eq. (2) was found for the case of an infinite plasma. Repeating essentially the same analysis, it is possible to generalize the calculation to account for sheath boundary conditions at the walls bounding the plasma. We shall not reproduce the algebra here, but merely present the results. If the walls are situated at $x = x_1$ and $x = x_2$, and the sheath potentials are $u_1 T$ and $u_2 T$, respectively, the particle and heat fluxes, j and q, become:

$$\begin{pmatrix} j \\ q \end{pmatrix} \equiv \int \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} \chi v f_0 \,\mathrm{d}^3 v = \begin{pmatrix} j_2 \\ q_2 \end{pmatrix} - \begin{pmatrix} j_1 \\ q_1 \end{pmatrix} - \frac{F(Z)}{\pi} \times \sqrt{\frac{8T(x)}{5(Z+1)m}} \int_{x_1}^{x_2} n(x') \begin{pmatrix} 1/T(x') \\ 1 \end{pmatrix} \sum_{i=0}^{2}\left[\begin{pmatrix} P_{-1/5,5/2}(g_i) \\ P_{0,7/2}(g_i) \end{pmatrix}\frac{\mathrm{d}T}{\mathrm{d}x'} + \begin{pmatrix} P_{-1/5,3/2}(g_i) \\ P_{0,5/2}(g_i) \end{pmatrix}\frac{\mathrm{d}e\phi}{\mathrm{d}x'}\right]\mathrm{d}x', \quad (4)$$

with $g_i(x, x') \equiv g_0(x, 2x_i - x')$, $i = 1,2$, and

$$P_{\alpha,\beta} \equiv P_{\alpha,\beta}(g) = \int_0^1 \xi^\alpha (1-\xi)^{-1/2}\mathrm{d}\xi \int_0^\infty \eta^\beta \exp\left[-\eta - \frac{g^2(x,x')}{\eta^5(1-\xi)}\right]\mathrm{d}\eta,$$

$$g_0^2(x,x') \equiv \frac{5(Z+1)}{4T^5(x')} \times \left[\int_x^{x'} 6\pi e^4 \Lambda n(x'') T^{1/2}(x'')\,\mathrm{d}x''\right]^2, \quad (5)$$

The Eqs. (4) and (5) are similar to those found in Ref. [7], apart from the terms involving g_1 and g_2, and the boundary terms

$$\begin{pmatrix} j_i \\ q_i \end{pmatrix} = \frac{4u_i^{5/2} n(x)}{\pi}\sqrt{\frac{10T(x)}{(Z+1)m}} \int_1^\infty \begin{pmatrix} 1 \\ \eta u_i T(x_i) \end{pmatrix} \eta^{3/2}\mathrm{d}\eta \int_1^\infty \exp\left[-u_i\xi\eta - \frac{g^2(x,x_i)}{u_i\eta^5(\xi^5-1)}\right]\frac{\mathrm{d}\xi}{\sqrt{\xi^5-1}}, \quad (6)$$

which result from the use of the sheath boundary condition. Note that these terms persist even if the density and temperature profiles are flat. The factor $F(Z)$ in Eq. (4)

was added a posteriori to make the expression for the heat flux valid for any ion charge Z in the limit of small mean-free path. $F(Z)$ is defined as

$$F(Z) \equiv 0.93 \frac{1 + 1.59Z^{-1} + 0.59Z^{-2}}{1 + 3.87Z^{-1} + 1.32Z^{-2}}, \tag{7}$$

and is almost, but not exactly, equal to unity for large Z, which reflects the circumstance that the high-energy expansion (Eq. (1)) of the collision operator is only approximate.

Because of the various approximations employed in their derivation, it is not immediately clear how accurate Eqs. (4)–(7) are. Additional justification by comparison with exact numerical solutions of the original Eq. (1), is therefore necessary. Such comparisons were published, e.g., in Refs. [6,7], and showed good agreement for slightly rippled temperature profiles as long as $\lambda/L < 0.1$. It is, however, important to make the comparison for profiles where the density and temperature vary *significantly*, since this is usually the case in applications. The original Eq. (1) has three independent variables, and is therefore nontrivial to solve, even numerically. However, for the particular density and temperature profiles

$$T(x) = T_0 x^{2/(1+2\alpha)},\ n(x) = n_0 x^{(3-2\alpha)/(1+2\alpha)},\ x > 0, \tag{8}$$

the ratio of the mean-free path to the profile scale-length $\gamma = \lambda/L = (T/2\pi e^4 \Lambda n)(\mathrm{d}T/\mathrm{d}x)$ is constant, and self-similar solutions are possible [11]. One of the independent variables can then be eliminated, making the equation easier to solve. We have constructed numerical solutions, adjusting the electric field so as to make the particle flux vanish, and calculated the ensuing normalized heat flux for various values of Z, γ, and α. In order to compare the findings with the corresponding results from the non-local Eqs. (4)–(7), we note first that the boundary terms vanish, since the temperature vanishes at the 'wall' $x = 0$. In fact, the wall is infinitely many mean-free paths removed from the plasma, and is therefore 'invisible' as far as the fast electrons are concerned. Next, we observe that the requirement that the particle flux vanish

$$j \propto \int \frac{n(x')}{T(x')} \left(P_{-1/5,5/2} \frac{\mathrm{d}T}{\mathrm{d}x'} - P_{-1/5,3/2} \frac{\mathrm{d}e\phi}{\mathrm{d}x'} \right) \mathrm{d}x' = 0 \tag{9}$$

implies that ϕ vary with x in the same manner as T, i.e., $\phi(x) = \phi_0 x^{2/(1+2\alpha)}$, and determines ϕ_0. We substitute this expression for $\phi(x)$ in the heat flux (Eq. (4)), and evaluate the integrals numerically and normalize to obtain the dimensionless heat flux $Q = q/[nT(2T/m)^{1/2}]$. Table 1 presents a comparison with the numerical results. The agreement is quite good, even for $Z = 1$, much better than one could have expected a priori.

In practice, it is cumbersome to solve the integral equation $j(\phi) = 0$ for the electrostatic potential $\phi(x)$. It is therefore desirable to have an expression for the heat flux that only involves $n(x)$ and $T(x)$, and not $\phi(x)$. In the limit of short mean-free path, $\partial/\partial x \to 0$, Eq. (9) implies that the electric field is $-\mathrm{d}e\phi/\mathrm{d}x = -5\,\mathrm{d}T/\mathrm{d}x$, and its effect on the heat flux, as found from Eq. (4), is to reduce it by a factor of 6,

$$q = q_2 - q_1 - \frac{F(Z)}{6\pi} \sqrt{\frac{8T(x)}{5(Z+1)m}} \int_{x_1}^{x_2} n(x') \times P_{0.7/2}[g(x, x')] \frac{\mathrm{d}T}{\mathrm{d}x'} \mathrm{d}x', \tag{10}$$

To investigate the accuracy of this approximation for longer mean-free paths, we compare its predictions for the profiles (Eq. (8)) with the above results in Table 1. Again, the agreement is mostly good, but now there is some discrepancy for large γ and α.

3. Fast electrons close to a wall in a homogeneous

Table 1
Comparison between the exact, numerically obtained, heat flux for the self-similar profiles (Eq. (8)), the analytical approximation (Eq. (4)) with the electric field adjusted to make the particle flux vanish, and the more approximate expression (Eq. (10))

		$\alpha = 4$			$\alpha = 8$		
		Eq. (4)	Eq. (10)	numerical	Eq. (4)	Eq. (10)	numerical
$Z = 1$	$\gamma = 0.001$	0.0043	0.0043	0.0041	0.0043	0.0043	0.0041
	$\gamma = 0.01$	0.043	0.044	0.045	0.041	0.046	0.038
	$\gamma = 0.1$	0.27	0.21	0.27	0.071	1.41	0.13
$Z = 3$	$\gamma = 0.001$	0.0028	0.0027	0.0027	0.0027	0.0028	0.0027
	$\gamma = 0.01$	0.028	0.029	0.029	0.026	0.028	0.026
	$\gamma = 0.1$	0.20	0.16	0.21	0.11	0.67	0.11
$Z = 10$	$\gamma = 0.001$	0.0013	0.0013	0.0012	0.0013	0.0013	0.0012
	$\gamma = 0.01$	0.013	0.013	0.013	0.012	0.013	0.012
	$\gamma = 0.1$	0.11	0.10	0.12	0.082	0.22	0.064

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If the temperature $T(y)$ is constant, a more rigorous analytical treatment of Eq. (2) is possible. In other words, it is possible to investigate *kinetically* the behavior of fast electrons close to a wall in a homogeneous, high Z, plasma. The purpose of such a calculation is to determine the electron flux as a function of the sheath potential. To our knowledge, earlier studies of this problem have merely stipulated the form of the incoming electron distribution function, e.g., to be a Maxwellian, never calculated it self-consistently.

For an analytical treatment in the limit $Z \gg 1$, $\varepsilon \gg T$ and $u \gg 1$, where $u = U/T$ is the normalized sheath potential, we introduce normalized variables, $w \equiv \varepsilon/T$, $z \equiv y/T^2$, $g \equiv f/n(m/2\pi T)^{3/2} e^{-\varepsilon/T}$, in which Eq. (2) takes the form

$$D[g] \equiv \frac{1}{w^3}\frac{\partial}{\partial w}\left(e^{-w}\frac{\partial g}{\partial w}\right) + e^{-w}\frac{\partial^2 g}{\partial z^2} = 0, \tag{11}$$

and the boundary conditions become $g(w=0,\ z)=1$ (since the distribution function approaches a Maxwellian at low energies), $g(w=\infty,\ z)<\infty$, $g(w>u,\ z=0)=0$, and $g_z'(w<u,\ z=0)=0$. It is now readily verified that the functional

$$J[g] \equiv \int_0^\infty dz \int_0^\infty e^{-w}\left[\left(\frac{\partial g}{dw}\right)^2 + w^3\left(\frac{\partial g}{\partial z}\right)^2\right] dw$$
$$= -\int_0^\infty \frac{\partial g}{\partial w}\bigg|_{w=0} dz - \int_0^\infty dz \int_0^\infty g D[g]\, dw, \tag{12}$$

defined for functions g satisfying the boundary conditions, assumes its minimum for the particular function g satisfying Eq. (11). But since

$$\int_0^\infty dz \int_0^\infty w^3 D[g]\, dw$$
$$= -\int_0^\infty \frac{\partial g}{\partial w}\bigg|_{w=0} dz - \int_u^\infty w^3 e^{-w}\frac{\partial g}{\partial z}\bigg|_{z=0} dw = 0, \tag{13}$$

it follows from a comparison with the particle flux associated with f_1 (Eq. (3)), that

$$j = -4n\left[\frac{T}{3\pi(Z+1)m}\right]^{1/2} \min J[g]. \tag{14}$$

In other words, the minimum value of the variational form $J[g]$ is equal to the particle flux to the wall within a multiplicative constant. A trial function which differs from the exact g by a small quantity $O(\delta_g)$ therefore gives a particle flux (Eq. (14)) with an error of only $O(\delta_g^2)$.

We now proceed to seek a suitable trial function. A naive approach would be to substitute a reasonable function satisfying the boundary conditions in Eq. (14), and evaluate the corresponding particle flux directly. In a more accurate method [12], instead of guessing the function g itself, one postulates the shape of its level curves, and then finds the optimum g that is constant on these curves. If we label the family of level curves by a parameter $0 < \eta < 1$, then $g(w,\ z) = g[\eta(w,\ z)]$, where the function $\eta(w,\ z)$ is still unknown. The functional $J[g]$ can then be written as

$$J[g] = \int_0^1 p(\eta)\left(\frac{\partial g}{\partial \eta}\right)^2 d\eta, \tag{15}$$

$$p(\eta) \equiv \int_0^\infty e^{-w(\eta,z)}\left[\left(\frac{\partial \eta}{\partial w}\right)^2 + w^3(\eta,\ z)\left(\frac{\partial \eta}{\partial z}\right)^2\right] \times \frac{dz}{|\partial\eta/\partial w|}. \tag{16}$$

The Euler–Lagrange equation for minimizing (Eq. (15)) now implies that $\partial g/\partial \eta = \text{const.}/p(\eta)$. Integrating this relation and requiring that $g(0)=0$, $g(1)=1$, gives

$$J[g] = \left(\int_0^1 d\eta/p(\eta)\right)^{-1} \tag{17}$$

If we provide a guess for the shape of the level curves $\eta(w,\ z)$, Eq. (17) now yields the minimum value of $J[g]$, given that g is constant on these curves.

A simple trial function $\eta(w,\ z)$ satisfying the boundary conditions\ is $w = (1-\eta)(u + z^k/\lambda\eta)$, where $k > 1$ and λ are free parameters which may be varied so as to minimize $J[g]$. With this choice of $\eta(w,\ z)$, the function $p(\eta)$ becomes

$$p(\eta) = \frac{(\lambda\eta)^{1/k}}{ku}\int_0^\infty \exp[(\eta-1)(u+s)] \times \left[s^{1/k-1} + \frac{k^2(1-\eta)^5 u^3}{(\lambda\eta)^{2/k}} s^{1-1/k}\right]\frac{ds}{1+s/\eta u} \tag{18}$$

to the lowest order in $1/u \ll 1$. This result is now to be inserted in Eq. (17), which should be minimized with respect to k and λ. The result of a numerical minimization is shown in Fig. 1. Dots representing the numerically

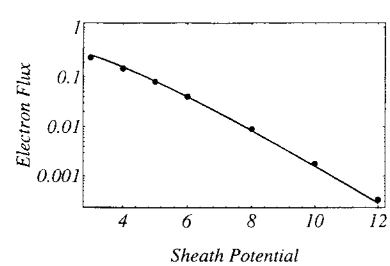

Fig. 1. The normalized electron particle flux $J[g]$ obtained variationally (dots), and the curve $J[g] = 1.1u^{3/2}\exp(-u)$ corresponding to Eq. (19).

obtained min $J[g]$ for various values of u are shown, along with the curve $1.1u^{3/2}\exp(-u)$. This curve apparently produces a close fit to the dots, so the flux of electrons to the wall (Eq. (14)) can be written as

$$j(u) \approx -1.4n\sqrt{\frac{T}{(Z+1)m}}\,u^{3/2}e^{-u}. \tag{19}$$

Equating this expression to the ion flux, determined by the Bohm sheath criterion, gives the sheath potential u in a high Z plasma.

The result (Eq. (19)) provides an opportunity to estimate the limits of the high Z approximation itself: Far from the wall the distribution is Maxwellian, and close to the wall the distribution function is devoid of high-energy particles travelling away from the wall. At intermediate distances from the wall, pitch-angle scattering fills in this void. The high-energy tail of the distribution is therefore depleted, and the particle flux should be smaller than that which would result from a Maxwellian, $j < j_M = n(T/2\pi m)^{1/2}e^{-u}$. Combined with Eq. (19), this inequality gives a lower bound for the ion charge Z,

$$Z > 12u^3, \tag{20}$$

which, unfortunately, is very high for realistic values of the sheath potential u. This constraint is in stark contrast with the fine agreement with similarity solutions we saw for *arbitrary* Z, and demonstrates the difficulties with the high Z approximation in the vicinity of walls.

4. Conclusions

The main result of this paper is the derivation of a non-local formula (Eqs. (4)–(7)), to be used in numerical edge codes, for the electron particle and heat fluxes in a plasma bounded by walls with Bohm sheath boundary conditions. The main simplifying assumption underlying the calculation is that of high ion charge, $Z \gg 1$. Even though this condition is not satisfied in a typical fusion edge plasma, the resulting heat flux appears to be surprisingly accurate if the plasma temperature falls off rapidly near the walls. Indeed, as long as $\lambda/L \leq 0.1$, the agreement with numerical results (Table 1) is excellent, even though the usual short mean-free path theory fails badly. The agreement is good even for $Z = 1$, which, strictly speaking, is beyond the validity of the assumptions made to simplify the problem.

Although the heat flux (Eqs. (4)–(7)) seems to be a good approximation and may be suitable for implementing in numerical edge computations, we observe that the non-local particle flux (which is important, e.g., for probe measurements [10]), calculated under similar assumptions, scales incorrectly in the limit of small density and temperature gradients. In a homogeneous background plasma, the particle flux to the wall is predicted by Eq. (6) to scale as

$$j \sim u^{5/2}\int_1^\infty \eta^{3/2}\,d\eta\int_1^\infty \exp(-\xi\eta u)(\xi^5-1)^{-1/2}\,d\xi$$
$$\approx (\pi/5)^{1/2}ue^{-u},\ u \gg 1, \tag{21}$$

in disagreement with Eq. (19).

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References

[1] L. Spitzer and R. Härm, Phys. Rev. 89 (1953) 977.
[2] S.I. Braginskii, in: Reviews of Plasma Physics, Vol. 1 (Interscience, New York, 1965) p. 205.
[3] A. Bell, R.G. Evans and D.J. Nicholas, Phys. Rev. Lett. 46 (1981) 243; A.R. Bell, Phys. Fluids 26 (1983) 279.
[4] J.F. Luciani, P. Mora and J. Virmont, Phys. Rev. Lett. 51 (1983) 1664; J.F. Luciani, P. Mora and A. Bendhib, Phys. Rev. Lett. 55 (1985) 2421.
[5] J.R. Albritton, E.A. Williams, I.B. Bernstein and K.P. Swartz, Phys. Rev. Lett. 57 (1986) 1887.
[6] E.M. Epperlein and R.W. Short, Phys. Fluids B 3 (1991) 3092.
[7] S.I. Krasheninnikov, Phys. Fluids B 5 (1993) 74.
[8] O.V. Batishchev, A.A. Batishcheva, P.J. Catto, S.I. Krasheninnikov and D.J. Sigmar, presented at 12th Int. Conf. on Plasma Surface Interactions, May, 1996, Saint-Raphaël, France.
[9] R.H. Cohen and T.D. Rognlien, Contrib. Plasma Phys. 34 (1994) 198.
[10] J.A. Wesson, Plasma Phys. Control. Fusion 37 (1995) 1459.
[11] S.I. Krasheninnikov, Sov. Phys. JETP 67 (1988) 2483.
[12] P.J. Catto and I.B. Bernstein, Phys. Fluids 24 (1981) 1900.